

Quadratically constrained quadratic programs on acyclic graphs with application to power flow

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Abstract

This paper proves that non-convex quadratically constrained quadratic programs can be solved in polynomial time when their underlying graph is acyclic, provided the constraints satisfy a certain technical condition. When this condition is not satisfied, we propose a heuristic to obtain a feasible point. We demonstrate this approach on optimal power flow problems over radial networks.

Index Terms

Optimal power flow, distribution circuits, radial networks, energy storage, SDP relaxation, minimum semidefinite rank.

I. INTRODUCTION

A quadratically constrained quadratic program (QCQP) is an optimization problem in which the objective function and the constraints are quadratic. In general, QCQPs are non-convex, and therefore lack computationally efficient solution methods. Many engineering problems including

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optimal power flow (OPF) can be represented as QCQPs with complex variables. The contribution of this paper is to expand the class of non-convex QCQPs for which globally optimal solutions can be guaranteed.

There is a large literature on optimal or approximate algorithms for QCQPs. One such method employs a convex semidefinite program that is a rank relaxation of the given QCQP. This is commonly referred to as semidefinite relaxation (SDR). Such semidefinite programs are solvable in polynomial time using interior-point methods [1]–[3]. In some instances, an optimal solution of the original QCQP can be recovered from an optimal solution of its SDR. In other cases, SDR provides a way to approximate the solution of a QCQP. Thus, SDR provides a computationally tractable way to approach QCQPs [4], [5]. For example, SDR has been applied to a variety of engineering problems such as MIMO antenna beam-forming [6]–[9], sensor network localization [10], principal component analysis [11] and stability analysis [12]. SDR has also been extensively used in systems and control theory applications [13], [14]. Several authors have investigated exact relaxations, e.g., [15], [16], while others have applied SDR-based approximation techniques to NP-hard combinatorial problems and non-convex QCQPs, e.g., [17]–[19]. The accuracy of these approximations has also been extensively studied, e.g., [20]–[22].

In this paper, we prove a sufficient condition under which QCQPs with underlying acyclic graph structures admit an efficient polynomial time solution using its SDR. We then apply our result to the optimal power flow (OPF) problem on radial networks. OPF is generally a non-convex, NP-hard problem that seeks to minimize some cost function, such as power loss, generation cost and/or user utilities, subject to engineering constraints. Since the original formulation of Carpentier in 1962 [23], various solution techniques have been used for this problem; we refer the reader to [24]–[28] for some surveys. OPF can be cast as a QCQP. The authors in [29], [30] propose to solve its SDR as an approximation. The authors in [31], instead, propose to solve its (convex) Lagrangian dual and provide a sufficient condition under which an optimal solution of the OPF can be recovered from a dual optimal solution. For IEEE test systems and other randomly generated circuits, these approaches have been shown to solve OPF optimally. Recently, OPF over radial networks has been of considerable interest. A checkable sufficient condition has been proved in [32]–[34] where an optimal solution of OPF can be recovered from an optimal solution of its SDR. This paper extends the previously known class of OPF problems that can be solved efficiently.

The paper is organized as follows. In Section II, we prove a sufficient condition for a non-convex QCQP over acyclic graphs to be solvable in polynomial time. In Section III, we apply this result to characterize a class of OPF problems over radial networks that can be solved efficiently. In Section IV we describe a heuristic method to obtain feasible solutions for QCQPs that do not meet these conditions and thus their optimum solution cannot be directly recovered by solving its SDR. We further apply this technique for the OPF problem and demonstrate through simulations that we can always find a near-optimal feasible point for OPF. We conclude in Section V.

II. QCQP'S AND SEMIDEFINITE RELAXATION

Consider the following QCQP with complex variable $x \in \mathbb{C}^n$.

Primal problem P :

$$\begin{aligned} & \underset{x \in \mathbb{C}^n}{\text{minimize}} && x^H C x \\ & \text{subject to:} && x^H C_k x \leq b_k, \quad k \in \mathcal{K}. \end{aligned}$$

where x^H denotes the conjugate transpose of x , C is either an $n \times n$ complex positive definite matrix (denoted as $C \succ 0$) or a positive semidefinite matrix (denoted as $C \succeq 0$), \mathcal{K} is a finite index set, C_k is an $n \times n$ complex Hermitian matrix and b_k is a scalar for each $k \in \mathcal{K}$.

If the matrices C_k , $k \in \mathcal{K}$ are positive semidefinite, then problem P is a convex program and can be solved in polynomial time [35], [36]. If, however, these matrices are not necessarily positive semidefinite, P is non-convex and NP-hard in general. The main result of this paper is the identification of a class of QCQPs that can be solved in polynomial time even though the matrices C_k , $k \in \mathcal{K}$ are not necessarily positive semidefinite. This result is applied in the next section to the optimal power flow problem on radial electric networks.

We begin with some notation. Let $[n] := \{1, 2, \dots, n\}$ and for any matrix H , let H_{ij} represent the element corresponding to the i^{th} row and the j^{th} column. Define a function \mathcal{G} from QCQP problems to undirected graphs as follows. For a QCQP problem P , the undirected graph $\mathcal{G}(P)$ has vertex set $[n]$ and the edge set defined as follows:

$$(i, j) \text{ is an edge in } \mathcal{G}(P) \iff (i \neq j) \text{ and } (C_{ij} \neq 0 \text{ or } [C_k]_{ij} \neq 0 \text{ for some } k \in \mathcal{K}) \quad (1)$$

Since the matrices C and C_k are Hermitian, the edges of the graph are undirected. Also, $\mathcal{G}(P)$ has no self-loops from a vertex to itself. We restrict attention to QCQP problems P for which the graph $\mathcal{G}(P)$ is a tree, i.e., it is connected and acyclic.

For any vector of real numbers a , let $a \gg 0$ denote that all elements of a are strictly positive. For any set of complex numbers $U = \{u_1, \dots, u_r\}$, the *relative interior of the convex hull* [36] of the set is defined as:

$$\{a_1 u_1 + a_2 u_2 + \dots + a_r u_r \in \mathbb{C} \mid a \gg 0 \text{ and } \sum_{\ell=1}^r a_\ell = 1\}. \quad (2)$$

We restrict our attention to QCQP problems for which the origin of the complex plane does not belong to the relative interior of the convex hull of the set $\{C_{ij}, [C_k]_{ij}, k \in \mathcal{K}\}$ for any edge (i, j) in $\mathcal{G}(P)$. For any edge (i, j) in $\mathcal{G}(P)$, consider the points C_{ij} and $[C_k]_{ij}, k \in \mathcal{K}$ on the complex plane. The convex hull of these points is either a single point, a line segment, or a convex polytope. The condition states that (a) if the hull is a single point, then that point is not the origin, (b) if the hull is a line segment, then the origin is either an extreme point of the line segment or the origin does not lie on the line segment, and (c) if the hull is a convex polytope then the origin is either outside or on the boundary of this polytope. This is illustrated in Figure 1.

We also limit the discussion to QCQPs for which the set of feasible solutions is bounded and has a strictly feasible point, i.e., there exists $x \in \mathbb{C}^n$ such that $x^H C_k x < b_k$ for all $k \in \mathcal{K}$.

To summarize, consider QCQPs that satisfy the following.

Condition 1: (a) $\mathcal{G}(P)$ is connected and acyclic.

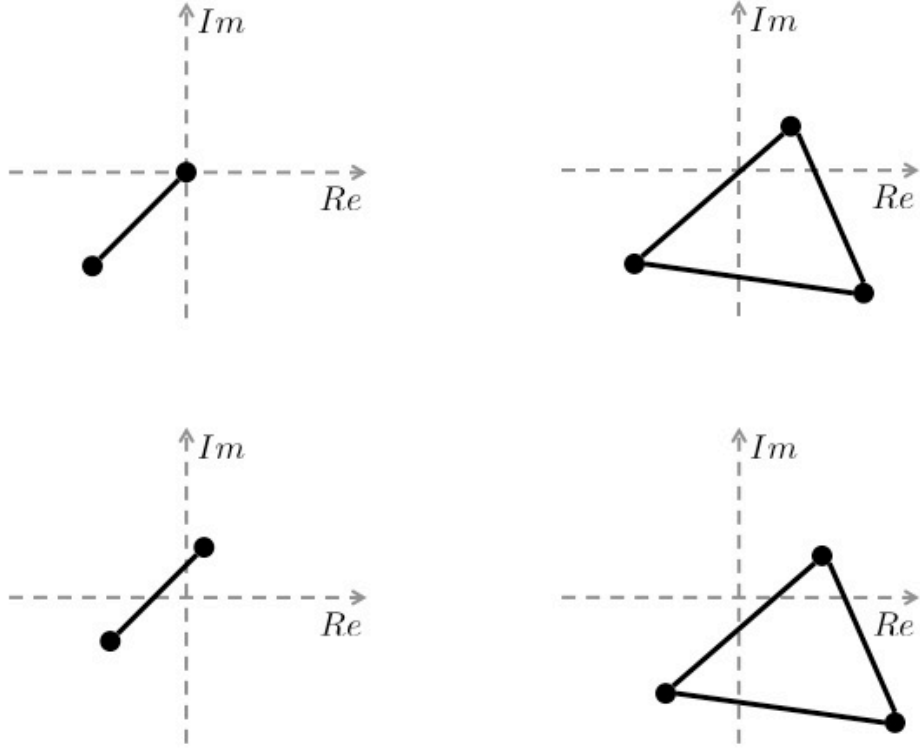
- (b) For any edge (i, j) in $\mathcal{G}(P)$, the origin is not in the relative interior of the convex hull of $\{C_{ij}, [C_k]_{ij}, k \in \mathcal{K}\}$.
- (c) The set of feasible solutions of P is bounded and has a strictly feasible point.

The main result of the paper is the following theorem and its application.

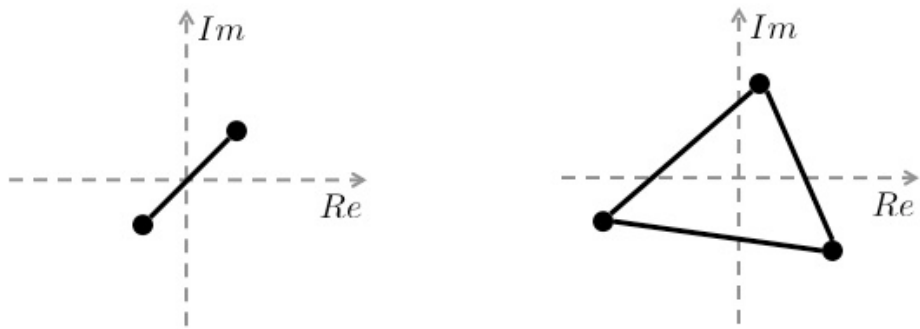
Theorem 1: All QCQPs P that satisfy condition 1 can be solved in polynomial time.

For a continuous optimization problem, we say it can be *solved in polynomial time* if given any $\zeta > 0$, there is an algorithm that finds a feasible solution to the optimization problem with an objective value within ζ of the theoretical optimum in polynomial time [5], [35], [36]. For QCQPs P that satisfy condition 1, Theorem 1 says that we can construct such a point in polynomial time.

To solve P , we use its convex relaxation that can be solved in polynomial time. The relaxation is said to be *exact*, if there exists an optimal solution of the relaxation that can be mapped to an optimal solution of P . An exact (convex) relaxation by itself does not however guarantee that P



(a) Origin does *not* belong to the relative interior of the convex hull of these points.



(b) Origin lies in the relative interior of the convex hull of these points.

Fig. 1: Convex hull of $\{C_{ij}, [C_k]_{ij}, k \in \mathcal{K}\}$ in condition 1(b).

can be solved in polynomial time. This is the case when the set of optimizers of the relaxation contains solutions that cannot be mapped to a feasible point in P and there may or may not be a polynomial time algorithm to find a correct optimum among that set of optimizers.¹ Theorem 1 asserts that when condition 1 holds, not only the convex relaxation of P is exact, but it can also be solved in polynomial time. See Remark 1 at the end of this section for more details.

In the remainder of this section, we prove Theorem 1 for the case where C is positive definite. The proof for the positive semidefinite case is presented in the appendix. Our proof requires the following result on Hermitian matrices, that is presented here and proved in the appendix.

Lemma 2: If $H_1 \succeq 0$ and $H_2 \succeq 0$ are two $n \times n$ matrices,

$$\text{tr}(H_1 H_2) \geq \rho_{\min}[H_1] \rho_{\max}[H_2].$$

where $\rho_{\min}[H]$ and $\rho_{\max}[H]$ respectively denote the minimum and maximum eigenvalues of any Hermitian matrix H .

A. Proof of Theorem 1 for $C \succ 0$:

For $C \succ 0$, we prove the result more generally by relaxing the condition that the feasible region of P is bounded. Consider the following semidefinite program RP where W is an $n \times n$ complex positive semidefinite matrix.

Relaxed Problem RP :

$$\begin{aligned} & \underset{W \succeq 0}{\text{minimize}} && \text{tr}(CW) \\ & \text{subject to:} && \text{tr}(C_k W) \leq b_k, \quad k \in \mathcal{K}. \end{aligned} \tag{3}$$

RP is a convex relaxation of P [5], [36]. Define p_* and r_* as the optimum values of the objective functions for problems P and RP respectively.

Lemma 3: p_* , r_* are finite and $p_* \geq r_*$. If W_* solves RP optimally and $\text{rank } W_* \leq 1$, then $p_* = r_*$ and problem P has an exact SDR.

Proof: The objective functions of P and RP are nonnegative and hence p_* and r_* are finite. Given any feasible solution x of P , $W := xx^H$ is a feasible solution of RP . Hence RP

¹ If every optimal solution of the relaxation can be mapped to a solution of P , then clearly P can be solved in polynomial time. This is sufficient but not a necessary condition for a polynomial time solution.

is feasible and $p_* \geq r_*$. If $\text{rank } W_* = 0$, then $W_* = 0$, and an optimal solution to P is $x_* = 0$, and therefore $r_* = p_*$. If $\text{rank } W_* = 1$ then W_* has a unique decomposition $W_* = x_* x_*^H$, where

$$r_* = \text{tr}(CW_*) = x_*^H C x_* = p_*.$$

■

Next, we show that there exists a finite W_* that solves RP optimally and has $\text{rank } W_* \leq 1$.

Let the Lagrange multipliers for the inequalities in (3) be $\lambda_k \geq 0$ for each $k \in \mathcal{K}$. Then the Lagrangian dual of RP is

Dual problem DP :

$$\begin{aligned} & \underset{\lambda \geq 0}{\text{maximize}} && - \sum_{k \in \mathcal{K}} \lambda_k b_k \\ & \text{subject to} && C + \sum_{k \in \mathcal{K}} \lambda_k C_k \succeq 0. \end{aligned}$$

For convenience, we introduce the $n \times n$ matrix $A(\lambda)$ defined as:

$$A(\lambda) := C + \sum_{k \in \mathcal{K}} \lambda_k C_k, \quad (4)$$

Define a function \mathcal{F} from Hermitian matrices to undirected graphs as follows. For any $n \times n$ Hermitian matrix H , the graph $\mathcal{F}(H)$ on the vertex set $[n]$ satisfies

$$(i, j) \text{ is an edge in } \mathcal{F}(H) \iff i \neq j \text{ and } H_{ij} \neq 0. \quad (5)$$

From the definitions of \mathcal{F} , \mathcal{G} and $A(\lambda)$, it follows that for any λ , $\mathcal{F}(A(\lambda))$ is a subgraph of $\mathcal{G}(P)$. For some values of λ however, edge (i, j) may exist in $\mathcal{G}(P)$ but not in $\mathcal{F}(A(\lambda))$; in this case $\mathcal{F}(A(\lambda))$ is acyclic but may not be connected, and hence it may be a forest of two or more disconnected trees rather than a single connected tree that spans all vertices in the graph. Now, we present a lemma about the connectedness of the graph $\mathcal{F}(A(\lambda))$.

Lemma 4: For all $\lambda \gg 0$, $\mathcal{F}(A(\lambda))$ is connected.

Proof: Consider any edge (i, j) in $\mathcal{G}(P)$. From condition 1, the origin is not in the relative interior of the convex hull of $\{C_{ij}, [C_k]_{ij}, k \in \mathcal{K}\}$. Using (4), we have $[A(\lambda)]_{ij} \neq 0$ and hence (i, j) is an edge of $\mathcal{F}(A(\lambda))$. Thus $\mathcal{F}(A(\lambda))$ is identical to $\mathcal{G}(P)$ which is a tree that spans all the vertices of the graph. ■

Next we characterize the relationship between the optimal points of RP and DP . Let d_* denote the optimal value of the objective function of DP .

Lemma 5: $r_* = d_*$ and RP/DP has a finite primal dual optimal point (W_*, λ_*) .

Proof: To prove this, we first show that DP is strictly feasible. At $\lambda = 0$, $\rho_{\min}[A(\lambda)] = \rho_{\min}[C] > 0$. For a sufficiently small $\lambda' \gg 0$, $\rho_{\min}[A(\lambda')] > 0$ and hence $A(\lambda') \succ 0$. This implies λ' is a strictly feasible point of DP . Also since P is assumed to be strictly feasible, RP is strictly feasible and it has a finite optimum r_* . The rest follows from Slater's condition [36]. ■
For convenience, define $A_* := A(\lambda_*)$. Now we turn our attention to the graph of A_* , i.e., $\mathcal{F}(A_*)$ to further analyze the primal dual optimum point (W_*, λ_*) of RP/DP .

Lemma 6: If $\mathcal{F}(A_*)$ is connected then $\text{rank } W_* \leq 1$.

Proof: We observe that $\text{rank } A_* \geq n-1$. This follows from a result in the literature [37], [38, Theorem 3.4] and [39, Corollary 3.9] that states that for any $n \times n$ positive semidefinite matrix H where the associated graph $\mathcal{F}(H)$ is a connected acyclic graph (i.e., a tree), $\text{rank } H \geq n-1$.

Next we show that $\text{rank } W_* \leq 1$. The complementary slackness condition for optimality of (W_*, λ_*) implies

$$\text{tr}(A_* W_*) = 0.$$

Let $W_* = \sum_i \rho_i w_i w_i^H$ be the spectral decomposition of W_* . Then,

$$\text{tr}(A_* W_*) = \sum_i \rho_i w_i^H A_* w_i = 0.$$

Since $A_* \succeq 0$, the eigenvectors w_i of W_* corresponding to nonzero eigenvalues ρ_i are all in the null space of A_* . The rank of A_* is at least $n-1$ and hence its null space has dimension at most 1, from which it follows that $\text{rank } W_* \leq 1$. ■

$\mathcal{F}(A_*)$ can be connected in one of two ways: (a) the origin of the complex plane lies strictly outside the convex hull of the set of points $\{C_{ij}, [C_k]_{ij} \neq 0, k \in \mathcal{K}\}$ for all edges (i, j) in $\mathcal{G}(P)$, or (b) $\lambda_* \gg 0$ (from lemma 4). In both cases, lemma 6 guarantees that $\text{rank } W_* \leq 1$.

If the origin lies on the boundary of the convex hull however, then $\mathcal{F}(A_*)$ may not be connected when λ_* is not element-wise strictly positive and therefore $\text{rank } W_* \leq 1$ may not hold. In that case, we cannot obtain an optimum solution of P from the optimum solution of RP . We deal with this case where $\mathcal{F}(A_*)$ is disconnected by using a perturbation [40], [41] of RP/DP so that $\mathcal{F}(A_*)$ is connected in the perturbed problem. Then we recover an optimal solution W_* for RP such that $\text{rank } W_* \leq 1$ from the perturbed problem.

Define the perturbed problems for parameter $\varepsilon > 0$:

Perturbed relaxed problem RP^ε :

$$\begin{aligned} & \underset{W \succeq 0}{\text{minimize}} && \text{tr}(CW) - \varepsilon \sum_{k \in \mathcal{K}} [b_k - \text{tr}(C_k W)] \\ & \text{subject to:} && \text{tr}(C_k W) \leq b_k, \quad k \in \mathcal{K}. \end{aligned}$$

Perturbed dual problem DP^ε :

$$\begin{aligned} & \underset{\lambda}{\text{maximize}} && - \sum_{k \in \mathcal{K}} \lambda_k b_k \\ & \text{subject to} && A(\lambda) \succeq 0, \quad \lambda_k \geq \varepsilon, \quad k \in \mathcal{K}. \end{aligned}$$

For any variable z in the original problem, let z^ε denote the corresponding variable in the perturbed problem with perturbation parameter ε .

Lemma 7: There exists a $\varepsilon_0 > 0$, such that for all ε in $(0, \varepsilon_0)$:

- 1) $r_*^\varepsilon = d_*^\varepsilon$ and $RP^\varepsilon/DP^\varepsilon$ has a finite primal dual optimal point $(W_*^\varepsilon, \lambda_*^\varepsilon)$.
- 2) $\mathcal{F}(A_*^\varepsilon)$ is connected and $\text{rank } W_*^\varepsilon \leq 1$.

Proof: We choose ε_0 as follows. For $\varepsilon > 0$, observe that

$$A(\varepsilon \mathbf{1}) = C + \varepsilon \sum_{k \in \mathcal{K}} C_k,$$

where $\mathbf{1}$ is a vector of all ones of appropriate size. Then $\rho_{\min}[A(\varepsilon \mathbf{1})]$ is a continuous function of ε that has the value $\rho_{\min}[C] > 0$ at $\varepsilon = 0$. Choose ε_0 sufficiently small such that $\rho_{\min}[A(\varepsilon \mathbf{1})]$ is strictly bounded away from 0, i.e.,

$$\min_{\varepsilon \in [0, \varepsilon_0]} \rho_{\min}[A(\varepsilon \mathbf{1})] > 0.$$

Consider any ε in $(0, \varepsilon_0)$. The feasible sets of RP and RP^ε are identical. Since $A(\varepsilon \mathbf{1}) \succ 0$ and $W \succeq 0$ for a feasible point W of RP (and RP^ε),

$$\text{tr}(CW) - \varepsilon \sum_{k \in \mathcal{K}} [b_k - \text{tr}(C_k W)] = \text{tr}[A(\varepsilon \mathbf{1}) W] - \varepsilon \sum_{k \in \mathcal{K}} b_k \geq -\varepsilon_0 \left| \sum_{k \in \mathcal{K}} b_k \right|$$

and hence r_*^ε is finite. $RP^\varepsilon/DP^\varepsilon$ are strictly feasible and r_*^ε is bounded below. The first part of lemma 7 then follows from Slater's condition [36].

To prove the second part of lemma 7, note that $\lambda_* \geq \varepsilon \mathbf{1} \gg 0$. Lemmas 4 and 6 applied to RP^ε proves the claim.

■

We have shown that for all ε in a nonempty interval $(0, \varepsilon_0)$, the optimal solution W_*^ε of RP^ε has rank at most 1. Now we analyze the behavior of W_*^ε as ε converges to zero. Let $\{\varepsilon_\ell\}_{\ell=1}^\infty$ be a decreasing sequence such that $\varepsilon_\ell \rightarrow 0$ as $\ell \rightarrow \infty$. Consider the sequence of matrices $\{W_*^{\varepsilon_\ell}\}_{\ell=1}^\infty$; every matrix in this sequence has rank at most 1. In the next lemma we show that this sequence has a convergent subsequence and the limit of this subsequence solves RP optimally.

Lemma 8: $\{W_*^{\varepsilon_\ell}\}_{\ell=1}^\infty$ has a convergent subsequence. The limit point \hat{W} of this subsequence solves RP optimally and satisfies $\text{rank } \hat{W} \leq 1$.

Proof: Consider any ε in $(0, \varepsilon_0)$. We first show that W_*^ε is bounded, independent of ε . For any W in the feasible set of RP (and RP^ε),

$$\text{tr}(CW) - \varepsilon \sum_{k \in \mathcal{K}} \underbrace{[b_k - \text{tr}(C_k W)]}_{\geq 0} \leq \text{tr}(CW).$$

Minimizing both sides over the feasible set of RP (and RP^ε), we have

$$r_*^\varepsilon \leq r_*, \quad (6)$$

that implies

$$\begin{aligned} \text{tr}[A(\varepsilon \mathbf{1}) W_*^\varepsilon] &= r_*^\varepsilon + \varepsilon \sum_{k \in \mathcal{K}} b_k \\ &\leq r_* + \varepsilon_0 \left| \sum_{k \in \mathcal{K}} b_k \right|. \end{aligned} \quad (7)$$

Also, Lemma 2 implies

$$\begin{aligned} \text{tr}[A(\varepsilon \mathbf{1}) W_*^\varepsilon] &\geq \rho_{\min}[A(\varepsilon \mathbf{1})] \rho_{\max}[W_*^\varepsilon] \\ &\geq \underbrace{\left[\min_{\varepsilon' \in [0, \varepsilon_0]} \rho_{\min}[A(\varepsilon' \mathbf{1})] \right]}_{>0 \text{ by construction.}} \rho_{\max}[W_*^\varepsilon]. \end{aligned} \quad (8)$$

Combining equations (7) and (8), we obtain a bound for $\rho_{\max}[W_*^\varepsilon]$, independent of ε :

$$\rho_{\max}[W_*^\varepsilon] \leq \frac{r_* + \varepsilon_0 \left| \sum_{k \in \mathcal{K}} b_k \right|}{\min_{\varepsilon' \in [0, \varepsilon_0]} \rho_{\min}[A(\varepsilon' \mathbf{1})]}.$$

Thus W_*^ε is bounded and $\text{rank } W_*^\varepsilon \leq 1$. Since the set of positive semidefinite matrices with rank at most 1 is closed [42], W_*^ε lies in a compact set and $\{W_*^{\varepsilon_\ell}\}_{\ell=1}^\infty$ has a convergent subsequence. Let the limit of this subsequence be \hat{W} . Then \hat{W} is feasible for RP and $\text{rank } \hat{W} \leq 1$. Next, we prove that $r_* = \text{tr}(C\hat{W})$ and hence \hat{W} solves RP optimally.

From (6),

$$r_*^\varepsilon = \text{tr}(CW_*^\varepsilon) - \varepsilon \sum_{k \in \mathcal{K}} [b_k - \text{tr}(C_k W_*^\varepsilon)] \leq r_*. \quad (9)$$

Taking limit over the convergent subsequence of $\{W_*^{\varepsilon_\ell}\}_{\ell=1}^\infty$, we get $\text{tr}(C\hat{W}) \leq r_*$. Also, r_* is the optimum value of RP and hence $r_* \leq \text{tr}(C\hat{W})$. This completes the proof of lemma 8. ■

So far we have shown that RP has a minimizer \hat{W} that satisfies $\text{rank } \hat{W} \leq 1$ and $p_* = r_*$, i.e., SDR of P is exact. But it is, in general, hard to guarantee that solving RP would yield the minimum rank optimizer if the set of optimizers of RP is non-unique. In that case, RP cannot be directly used to solve P in polynomial time. In what follows, we present an algorithm to solve P in polynomial time.

First, solve RP in polynomial time to obtain r_* . If the associated optimizer W_* has rank at most 1, then construct x_* from W_* as in lemma 3. We have then found x_* in polynomial time that solves P optimally. If however $\text{rank } W_* > 1$, choose ε_0 as given above and solve RP^{ε_0} in polynomial time. For any ε in $(0, \varepsilon_0)$,

$$r_*^\varepsilon = \text{tr}(CW_*^\varepsilon) - \varepsilon \sum_{k \in \mathcal{K}} [b_k - \text{tr}(C_k W_*^\varepsilon)] \leq r_* \leq \text{tr}(CW_*^\varepsilon), \quad (10)$$

where the first inequality follows from (6) and the second one follows from the fact that r_* is the optimum value of RP . Also, comparing the objective function values of RP^ε and RP^{ε_0} at W_*^ε and $W_*^{\varepsilon_0}$ respectively, we have

$$\sum_{k \in \mathcal{K}} [b_k - \text{tr}(C_k W_*^\varepsilon)] \leq \sum_{k \in \mathcal{K}} [b_k - \text{tr}(C_k W_*^{\varepsilon_0})]. \quad (11)$$

Combining (10) and (11), we have

$$|r_* - \text{tr}(CW_*^\varepsilon)| \leq \varepsilon \sum_{k \in \mathcal{K}} [b_k - \text{tr}(C_k W_*^{\varepsilon_0})].$$

Given any $\zeta > 0$, choose ε in $(0, \varepsilon_0)$ such that $\sum_{k \in \mathcal{K}} [b_k - \text{tr}(C_k W_*^{\varepsilon_0})] \leq \zeta$. Now solve RP^ε in polynomial time to get W_*^ε that satisfies $\text{rank } W_*^\varepsilon \leq 1$ and compute x_*^ε from it. Then x_*^ε is a feasible point of P and

$$p_* \leq (x_*^\varepsilon)^\mathcal{H} C (x_*^\varepsilon) \leq p_* + \zeta.$$

Also, we have computed x_*^ε in polynomial time. This completes the proof of Theorem 1 for the case where C is positive definite. We remark that a perturbation by an arbitrary small $\varepsilon > 0$

can be represented by treating each perturbed scalar variable as a pair $[a, a']$ to represent $a + \varepsilon a'$ when solving RP using any standard polynomial time algorithm like the interior-point method [1]–[3], where the pairs $[a, a']$ are ordered lexicographically [43].

The proof extending the theorem to the case where C is positive semidefinite is given in the appendix.

Remark 1: The strict feasibility of P in Condition 1 is required to solve P in polynomial time. If we relax that constraint, it can be shown that there still exists a positive semidefinite matrix W_* that solves RP optimally and $\text{rank } W_* \leq 1$, i.e., P has an exact SDR. There might also be other optimal solutions of RP that do not satisfy the rank condition and hence cannot be mapped to an optimal solution of P . Solving for a low-rank optimizer arbitrarily closely in polynomial time is hard to guarantee and is a direction for future work.

III. OPTIMAL POWER FLOW: AN APPLICATION

In this section, we apply the results of Section II to the optimal power flow (OPF) problem. We start by summarizing some of the recent results on OPF relaxations in Section III-A. In Section III-B we formulate OPF as a QCQP. In Section III-C we restrict our attention to OPF over radial networks, which are the networks commonly found in distribution circuits, and use Theorem 1 to characterize a set of conditions under which OPF can be solved efficiently.

A. Prior work

As previously discussed, OPF can be cast as a QCQP. Various non-linear programming techniques have been applied to the resulting nonconvex problem, e.g., in [44]–[46]. An SDR for OPF has been explored in [29], [30] and their simulations indicate that the SDR provides an exact solution of original OPF for many of the IEEE test systems [47]. The authors in [31], [48] propose to solve the convex Lagrangian dual of OPF and derive a sufficient condition under which an optimal solution of OPF can be recovered from an optimal dual solution. Though SDR recovers a solution to OPF on IEEE test systems, it does not work on all problem instances. Such limitations have been most recently reported in [49], though the nonconvexity of power flow solutions have been studied much earlier, e.g., in [33], [50]–[52].

Recently, a series of work has explored a class of problems where such limitations do not apply due to the network topology. It has been independently reported in [32]–[34] that the SDR

of OPF is exact for radial networks provided certain conditions on the power flow constraints are satisfied. A different approach to OPF has been explored using the branch flow model, first introduced in [53], [54]. While [55] studies a linear approximation of this model, various relaxations based on second-order cone programming (SOCP) have been proposed in [56]–[59]. Authors in [57]–[59] prove that this relaxation is exact for radial networks when there are no upper bounds on loads, or when there are no upper bounds on voltage magnitudes.

Motivated by the results in [57], a more general branch flow model is introduced in [60] for the power flow analysis and optimization for both radial and meshed networks. The precise relationships between the SOCP relaxations and the SDR for the OPF problem has been recently identified in [61].

B. Problem Formulation

Consider a power system network with n nodes (buses). The admittance-to-ground at bus i is y_{ii} and the admittance of the line between connected nodes i and j (denoted by $i \sim j$) is $y_{ij} = g_{ij} - \mathbf{i}b_{ij}$. We assume both $g_{ij} > 0$ and $b_{ij} > 0$, i.e., the lines are resistive and inductive. Define the corresponding $n \times n$ admittance matrix Y as

$$Y_{ij} = \begin{cases} y_{ii} + \sum_{j \sim i} y_{ij}, & \text{if } i = j, \\ -y_{ij}, & \text{if } i \neq j \text{ and } i \sim j, \\ 0 & \text{otherwise.} \end{cases} \quad (12)$$

Remark 2: Y is symmetric but not necessarily Hermitian.

The remaining circuit parameters and their relations are defined as follows.

- V and I are n -dimensional complex voltage and current vectors, where V_k , I_k denote the voltage and the injection current at bus $k \in [n]$ respectively. The voltage magnitude at each bus is bounded as

$$0 < \underline{W}_k \leq |V_k|^2 \leq \overline{W}_k, \quad k \in [n].$$

- $S = P + \mathbf{i}Q$ is the n -dimensional complex power vector, where P and Q respectively denote the real and reactive powers and

$$S_k = P_k + \mathbf{i}Q_k = V_k I_k^{\mathcal{H}}, \quad k \in [n]. \quad (13)$$

- P_k^D and Q_k^D are the real and reactive power demands at bus $k \in [n]$, which are assumed to be fixed and given.
- P_k^G and Q_k^G are the real and reactive power generation at bus k . They are decision variables that satisfy the constraints $\underline{P}_k^G \leq P_k^G \leq \overline{P}_k^G$ and $\underline{Q}_k^G \leq Q_k^G \leq \overline{Q}_k^G$.

Power balance at each bus $k \in [n]$ requires $P_k^G = P_k^D + P_k$ and $Q_k^G = Q_k^D + Q_k$, which leads us to define

$$\begin{aligned}\underline{P}_k &:= \underline{P}_k^G - P_k^D, & \overline{P}_k &:= \overline{P}_k^G - P_k^D \\ \underline{Q}_k &:= \underline{Q}_k^G - Q_k^D, & \overline{Q}_k &:= \overline{Q}_k^G - Q_k^D.\end{aligned}$$

The power injections at each bus $k \in [n]$ are then bounded as

$$\underline{P}_k \leq P_k \leq \overline{P}_k, \quad \underline{Q}_k \leq Q_k \leq \overline{Q}_k.$$

The branch power flows and their limits are defined as follows.

- $S_{ij} = P_{ij} + \mathbf{i}Q_{ij}$ is the sending-end complex power flow from node i to node j , where P_{ij} and Q_{ij} are the real and reactive power flows respectively. The real power flows are constrained as $|P_{ij}| \leq \overline{F}_{ij}$ where \overline{F}_{ij} is the line-flow limit between nodes i and j and $\overline{F}_{ij} = \overline{F}_{ji}$.
- $L_{ij} = P_{ij} + P_{ji}$ is the power loss over the line between nodes i and j , satisfying $L_{ij} \leq \overline{L}_{ij}$ where \overline{L}_{ij} is the thermal line limit and $\overline{L}_{ij} = \overline{L}_{ji}$. Also, observe that since $L_{ij} \geq 0$, we have $|P_{ij}| \leq \overline{F}_{ij}, |P_{ji}| \leq \overline{F}_{ji}$ if and only if $P_{ij} \leq \overline{F}_{ij}, P_{ji} \leq \overline{F}_{ji}$.

Let $J_k = e_k e_k^H$ where e_k is the k -th canonical basis vector in \mathbb{C}^n . Define $Y_k := e_k e_k^H Y$.

Substituting these expressions into (13) yields

$$\begin{aligned}S_k &= e_k^H V I^H e_k = \text{tr}(V V^H (Y^H e_k e_k^H)) = V^H Y_k^H V \\ &= \left(V^H \underbrace{\left(\frac{Y_k^H + Y_k}{2} \right)}_{=: \Phi_k} V \right) + \mathbf{i} \left(V^H \underbrace{\left(\frac{Y_k^H - Y_k}{2\mathbf{i}} \right)}_{=: \Psi_k} V \right),\end{aligned}\tag{14}$$

where Φ_k and Ψ_k are Hermitian matrices. Thus, the two quantities $V^H \Phi_k V$ and $V^H \Psi_k V$ are real numbers; moreover

$$P_k = V^H \Phi_k V, \quad Q_k = V^H \Psi_k V.$$

The real power flow from i to j can be expressed as a quadratic form as follows.

$$P_{ij} = \text{Re}\{V_i(V_i - V_j)^{\mathcal{H}}y_{ij}^{\mathcal{H}}\} = V^{\mathcal{H}}M^{ij}V, \quad (15)$$

where M^{ij} is an $n \times n$ Hermitian matrix. Further details of the OPF problem formulation are provided in the appendix.

The thermal loss of the line connecting buses i and j is

$$L_{ij} = L_{ji} = P_{ij} + P_{ji} = V^{\mathcal{H}}T^{ij}V \quad (16)$$

where $T^{ij} = T^{ji} := M^{ij} + M^{ji} \succeq 0$.

For a Hermitian positive semidefinite $n \times n$ matrix C , we have

Optimal power flow problem OPF:

$$\underset{V \in \mathbb{C}^n}{\text{minimize}} \quad V^{\mathcal{H}}CV$$

subject to:

$$\underline{P}_k \leq V^{\mathcal{H}}\Phi_k V \leq \overline{P}_k, \quad k \in [n], \quad (17a)$$

$$\underline{Q}_k \leq V^{\mathcal{H}}\Psi_k V \leq \overline{Q}_k, \quad k \in [n], \quad (17b)$$

$$\underline{W}_k \leq V^{\mathcal{H}}J_k V \leq \overline{W}_k, \quad k \in [n], \quad (17c)$$

$$V^{\mathcal{H}}M^{ij}V \leq \overline{F}_{ij}, \quad i \sim j, \quad (17d)$$

$$V^{\mathcal{H}}T^{ij}V \leq \overline{L}_{ij}, \quad i \sim j, \quad (17e)$$

where (17a)–(17e) are respectively constraints on the real and reactive powers, the voltage magnitudes, the line flows and thermal losses. Note that since $T^{ij} \succeq 0$, (16) implies that $P_{ij} + P_{ji} \geq 0$. This means that (17d) holds if and only if $|P_{ij}| \leq \overline{F}_{ij}$, i.e., (17d) bounds the line flows on both ends.

We do not include line-flow constraints that impose an upper bound on the apparent power $\sqrt{P_{ij}^2 + Q_{ij}^2}$ on each branch $i \sim j$. These constraints are not quadratic in voltages and hence beyond the scope of our model.

Remark 3 (Objective Functions): We consider different optimality criteria by changing C as follows:

- Voltages: $C = \mathcal{I}_{n \times n}$ (identity matrix) where we aim to minimize $\|V\|^2 = \sum_k |V_k|^2$.

- Power loss: $C = (Y + Y^H)/2$ where we aim to minimize $\sum_i g_{ii}|V_i|^2 + \sum_{i < j} g_{ij}|V_i - V_j|^2$.
- Production costs: $C = \sum_k c_k \Phi_k$ where we aim to minimize $\sum_k c_k P_k^G$, $c_k \geq 0$.

We assume C is positive semidefinite.

C. Semidefinite relaxation of OPF over radial networks

Assume hereafter that OPF is feasible. To conform to the notations of Section II, replace the constraint in (17a) by the equivalent constraints

$$\begin{aligned} V^H[\Phi_k]V &\leq \overline{P}_k, \quad k \in [n] \\ V^H[-\Phi_k]V &\leq -\underline{P}_k, \quad k \in [n]. \end{aligned}$$

Similarly rewrite (17b) and (17c). Then the set of matrices $\{C_k, k \in \mathcal{K}\}$ and the set of scalars $\{b_k, k \in \mathcal{K}\}$ in OPF are defined as

$$\{C_k, k \in \mathcal{K}\} := \{\Phi_k, -\Phi_k, \Psi_k, -\Psi_k, J_k, -J_k, k \in [n]\} \cup \{M^{ij}, T^{ij}, i \sim j\} \quad (18)$$

$$\{b_k, k \in \mathcal{K}\} := \{\overline{P}_k, -\underline{P}_k, \overline{Q}_k, -\underline{Q}_k, \overline{W}_k, -\underline{W}_k, k \in [n]\} \cup \{\overline{F}^{ij}, \overline{L}^{ij}, i \sim j\} \quad (19)$$

We limit the discussion to OPF instances where the graph of the power network is a tree \mathcal{T} on n nodes. It can be checked that the graph of the problem OPF satisfies

$$\mathcal{G}(OPF) = \mathcal{T}. \quad (20)$$

Thus condition 1(a) holds for OPF over \mathcal{T} . In general, condition 1(b) does not hold for OPF over \mathcal{T} . To illustrate this point, consider an edge (i, j) in \mathcal{T} . The admittance of the line joining buses i and j is $g_{ij} - \mathbf{i}b_{ij}$. Then $[C_k]_{ij}, k \in \mathcal{K}$ are given as (details are in the appendix):

- (a) $[\Phi_i]_{ij} = -g_{ij}/2 + \mathbf{i}b_{ij}/2,$
- (b) $[\Phi_j]_{ij} = -g_{ij}/2 - \mathbf{i}b_{ij}/2,$
- (c) $[\Psi_i]_{ij} = -b_{ij}/2 - \mathbf{i}g_{ij}/2,$
- (d) $[\Psi_j]_{ij} = -b_{ij}/2 + \mathbf{i}g_{ij}/2,$
- (e) $[M^{ij}]_{ij} = -g_{ij}/2 + \mathbf{i}b_{ij}/2,$
- (f) $[M^{ji}]_{ij} = -g_{ij}/2 - \mathbf{i}b_{ij}/2,$
- (g) $[T^{ij}]_{ij} = [T^{ji}]_{ij} = -g_{ij}.$

For the objective functions considered, we have

- (a) Voltages: $C_{ij} = 0,$

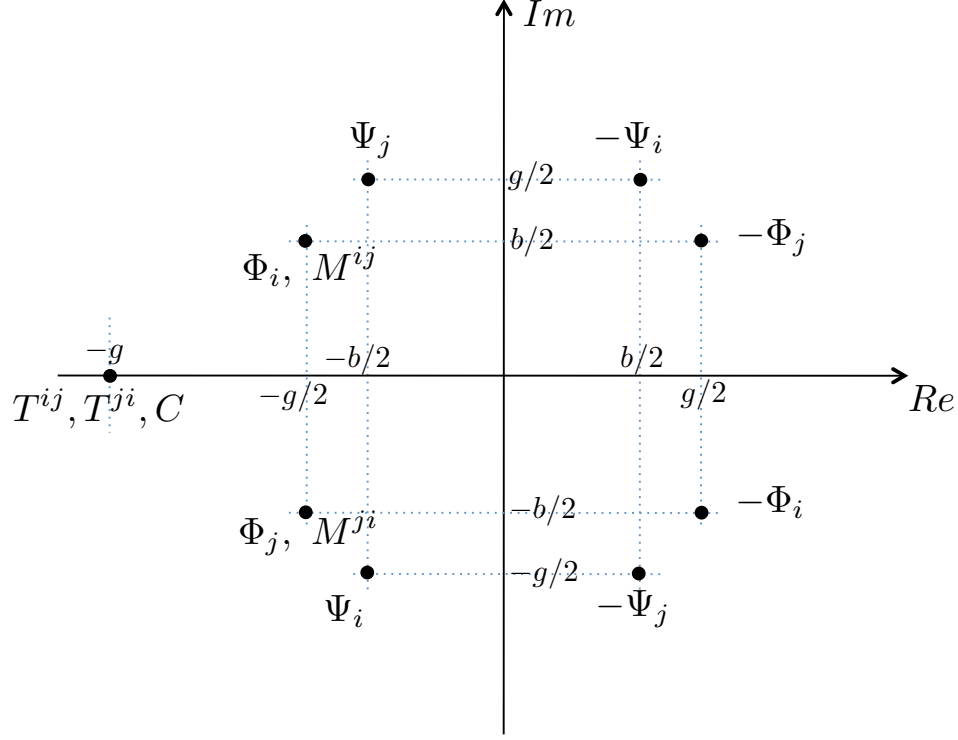


Fig. 2: C_{ij} and non-zero $([C_k]_{ij}, k \in \mathcal{K})$ on the complex plane for OPF for a fixed line (i, j) in tree \mathcal{T} .

(b) Power loss: $C_{ij} = -g_{ij}$,

(c) Production costs: $C_{ij} = -g_{ij}(c_i + c_j)/2 + \mathbf{i}b_{ij}(c_i - c_j)/2$.

For the purpose of this discussion, consider power-loss minimization as the objective, i.e., $C_{ij} = -g_{ij}$. Also, assume $g_{ij} > b_{ij} > 0$. We plot the non-zero (i, j) -th entries of the matrices $C_k, k \in \mathcal{K}$ and C , on the complex plane in Figure 2 and label each point with its corresponding matrix. Clearly if we consider all the points $[C_k]_{ij}, k \in \mathcal{K}$ and C_{ij} , the origin of the complex plane lies in the relative interior of the convex hull of these points, i.e., condition 1(b) does not hold.

To apply Theorem 1 to OPF , consider the index-set $\tilde{\mathcal{K}} \subseteq \mathcal{K}$ so that condition 1(b) holds for $C_k, k \in \tilde{\mathcal{K}}$ and C . This corresponds to removing certain inequalities in OPF , i.e., $b_k = +\infty$ for $k \in \mathcal{K} \setminus \tilde{\mathcal{K}}$. For example, removing $-\Phi_j$ from the set $\{C_k, k \in \mathcal{K}\}$ corresponds to setting $\underline{P}_j = -\infty$.

Condition 1(c) requires that the feasible set of OPF be bounded. This is always the case

when \overline{W}_k is finite for all buses $k \in [n]$. We discuss the technical condition of strict feasibility in remark 4.

Theorem 9: For $\tilde{\mathcal{K}} \subseteq \mathcal{K}$, suppose condition 1 holds for OPF with $C_k, k \in \tilde{\mathcal{K}}$ and C . Then OPF can be solved in polynomial time.

Remark 4: When OPF is strictly feasible, it can be solved in polynomial time. When strict feasibility does not hold, it follows from remark 1 that OPF still has an exact SDR but we do not guarantee a polynomial time solution.

We explore, through examples, some constraint patterns for OPF over radial networks where a polynomial time solution (for strictly feasible OPF instances) or an exact SDR (for OPFs that may not be strictly feasible) is guaranteed.

Example 1: In Figure 2, consider the (i, j) -th elements of the following set of matrices:

$$\{\Phi_i, \Phi_j, \Psi_i, \Psi_j, -\Psi_i, M^{ij}, M^{ji}, T^{ij} = T^{ji}, C\}.$$

The origin of the complex plane lies on the boundary of the convex hull of these points but not in its relative interior. With this set of points, associate a constraint pattern defined as follows. For any point in the diagram that is not a part of this set, the inequality associated with that matrix is removed from OPF . For example, the matrices $-\Phi_j$, $-\Phi_i$ and $-\Psi_j$ do not feature on the list of points. Hence,

$$\underline{P}_j = \underline{P}_i = \underline{Q}_j = -\infty. \quad (21)$$

This can be generalized to a constraint pattern over \mathcal{T} by removing the lower bounds on real powers at all nodes and the lower bounds on reactive powers at alternate nodes.

Example 2: Suppose $\underline{P}_k = \underline{Q}_k = -\infty$ for all nodes k in \mathcal{T} . This corresponds to considering points only on the left-half plane in Figure 2 for all edges (i, j) in \mathcal{T} . Clearly, condition 1(b) holds in this case. In Figure 2, we assume $g_{ij} > b_{ij} > 0$. Regardless of the ordering between g_{ij} and b_{ij} for edges (i, j) in \mathcal{T} , the set of points considered in this constraint pattern always lies in the left half of the complex plane.

Removing the lower bounds in real and reactive power can be interpreted as load over-satisfaction, i.e., the real and reactive powers supplied to a node k can be greater than their respective real and reactive power demands P_k^D and Q_k^D . OPF on a radial network with load over-satisfaction can be solved efficiently. This result has been reported in [32]–[34].

Example 3: Consider voltage minimization, i.e., $C = \mathcal{I}_{n \times n}$. In Figure 2, consider the (i, j) -th entries of the following set of matrices:

$$\{-\Phi_i, \Phi_j, -\Phi_j, \Psi_i, -\Psi_j, C\}.$$

The origin of the complex plane is again on the boundary of the convex hull of these points. The constraint pattern associated with this set of points is

$$\overline{P}_i = \overline{Q}_j = \overline{L}_{ij} = \overline{L}_{ji} = \overline{F}_{ij} = \overline{F}_{ji} = +\infty, \quad \text{and} \quad \underline{Q}_i = -\infty.$$

This constraint pattern is consistent with condition 1(b) over the edge (i, j) and we can construct a constraint pattern for the OPF problem.

IV. NUMERICAL EXAMPLES

A. Numerical techniques

In Section II, we have identified conditions under which an SDR of a QCQP over an acyclic connected graph can be used to solve the nonconvex problem P efficiently. When these conditions are not satisfied, the QCQP may not be polynomial time computable or have an exact SDR, i.e. RP yields an optimal W_* with rank $W_* > 1$. In that case, W_* cannot be mapped to an optimal x_* for the problem P . In this section we propose a method to construct a feasible solution \tilde{x} for P using such an optimal W_* of RP . The following relation characterizes the relationship between the optimal solution of P and its value at \tilde{x} :

$$\begin{aligned} \text{objective value of } RP \text{ at } W_* &\leq \text{optimum objective value of } P \\ &\leq \text{objective value of } P \text{ at } \tilde{x}. \end{aligned} \quad (22)$$

In many practical problems where rank $W_* > 1$, the principal eigenvalue of W_* is orders of magnitude greater than the other eigenvalues. We use the principal eigenvector to search for a “nearby” feasible point of P as follows. Let $w_* \in \mathbb{C}^n$ be the principal eigenvector of W_* and define the starting point x_0 of the algorithm as

$$x_0 := w_* \sqrt{\text{tr}(CW_*)}.$$

This scaling ensures that the objective value at x_0 is the optimum objective value of RP at W_* . If x_0 satisfies all constraints in P , then the algorithm ends with $\tilde{x} = x_0$. Otherwise, we construct a sequence of points (x_1, x_2, \dots) where x_{m+1} is constructed from x_m as follows.

- 1) For $k \in \mathcal{K}$, linearize the function $f_k(x) = x^H C_k x$ around the point x_m and call this function $f_k^{(m)}(x)$, i.e.,

$$f_k^{(m)}(x) = x_m^H C_k x_m + 2 \operatorname{Re} [x_m^H C_k (x - x_m)].$$

- 2) For $k \in \mathcal{K}$, define

$$s_k^{(m)}(x) := \begin{cases} \underline{b}_k - f_k^{(m)}(x), & \text{if } f_k^{(m)}(x) \leq \underline{b}_k, \\ 0 & \text{if } \underline{b}_k \leq f_k^{(m)}(x) \leq \bar{b}_k, \\ f_k^{(m)}(x) - \bar{b}_k, & \text{if } \bar{b}_k \leq f_k^{(m)}(x). \end{cases}$$

We can interpret $s_k^{(m)}(x)$ as the amount by which the linearized function $f_k^{(m)}$ violates the inequality constraint $\underline{b}_k \leq f_k^{(m)}(x) \leq \bar{b}_k$.

- 3) Compute x_{m+1} using

$$\begin{aligned} x_{m+1} = \arg \min_{x \in \mathbb{C}^n} \quad & \sum_{k \in \mathcal{K}} [s_k^{(m)}(x)]^2 \\ \text{subject to:} \quad & \|x - x_m\|_1 \leq \gamma, \end{aligned}$$

where $\|\cdot\|_1$ denotes the ℓ^1 norm and γ is the maximum allowable step-size. This is a parameter for the algorithm and should be chosen such that the linearization $f_k^{(m)}(x)$ is a reasonably good approximation of the quadratic form $f_k(x)$ for all $k \in \mathcal{K}$ in the ℓ^1 ball centered around x_m with radius γ .

- 4) If x_{m+1} satisfies all constraints in P , then the algorithm ends with $\tilde{x} = x_{m+1}$.

This heuristic approach either ends at a feasible point \tilde{x} of P or it fails to produce one within a fixed number of iterations. In the next section, we show that this technique performs quite well for numerical *OPF* examples where the SDR yields an optimal solution W_* with rank more than 1.

B. *OPF* test examples

The SDR of *OPF* and the techniques described in section III are illustrated on a sample distribution circuit from Southern California and randomly generated radial circuits. The semidefinite program is solved in MATLAB using YALMIP [62]. If the solution yielded W_* such that $\operatorname{rank} W_* \leq 1$ then the optimal voltage profile (V_*) to the *OPF* problem is calculated from $W_* = (V_*)(V_*)^H$. If the optimal W_* does not satisfy the rank condition, the heuristic approach

described above is used to find a feasible point of OPF . The feasible point obtained may not be optimal for the original problem, so we characterized its sub-optimality by defining the following quantity.

$$\eta := \frac{\text{Objective value at heuristically reached feasible point}}{\text{Objective value at optimal point of relaxed problem}} - 1.$$

Smaller values of η indicate that the feasible point obtained using the algorithm is close to the theoretical optimum of OPF .

Throughout this section, let $y = (a, b)$ denote a y drawn from a uniform distribution over the interval $[a, b]$. Using this notation, we describe the test systems used for simulations.

- 1) *SoCal Distribution Circuit*: The sample industrial distribution system in Southern California has been previously reported in [57]. It has a peak load of approximately 11.3 MW and installed PV generation capacity of 6.4MW. We modified this circuit by removing the 30MW load at the substation bus (that represented other distribution circuits fed by the same substation) and simulated it with the parameters provided in Table I. To scale the problem correctly, the problem was cast in per unit (p.u.) quantities using base values given in Table I.
- 2) *Random Test Circuits*: These circuits are generated using parameters typical of sparsely loaded rural circuits, as detailed in [63] and employed (with suitable modifications) in [64], [65]. Around 15-60% of the nodes are assumed to have 2 kW of PV capacity. The remaining parameters of these systems are described in Table I.

The tests are run with both voltage and power-loss minimization as objective functions. The optimization results are summarized in Table II. For power-loss minimization, we always obtain a rank 1 optimal W_* .

For voltage minimization, however, we obtain optimal solutions that violate the rank condition. In these cases, the heuristic approach is used to find a feasible point of OPF . We construct the solution based on the complex voltage $V_k = |V_k|e^{i\theta_k}$ at bus $k \in [n]$. For the heuristic algorithm, define

$$x := (|V_2|, |V_3|, \dots, |V_n|, \theta_2, \theta_3, \dots, \theta_n),$$

and set the parameter $\gamma = +\infty$. In the examples studied, this approach always yields a feasible point within 5 iterations. From Table II, the values obtained for η indicate that our algorithm

Test system	SoCal distribution circuit	Random radial networks
Number of nodes (n)	47	50-150
Line impedances $(y_{ij})^{-1}$	[57, Table 1]	$(0.33 + \mathbf{i}0.38)\Omega/km$, length = $(0.2km, 0.3km)$
Voltage limits $\sqrt{W_k}, \sqrt{W_k}$	1 ± 0.05 p.u. at all nodes.	1 ± 0.05 p.u. at all nodes.
Real power demand P_k^D	[57, Table 1]	$(0, 4.5kW)$
Reactive power demand Q_k^D	Computed with $p.f. = (0.80, 0.98)$ lagging	$(0.2P_k^D, 0.3P_k^D)$
Real power gen. limits $\bar{P}_k^G, \underline{P}_k^G$	PV nodes: $\bar{P}_k^G = (0.2, 1.0)$ times capacity, Substation node: $\bar{P}_1^G = 10MW$. At all nodes, $\underline{P}_k^G = 0$.	PV nodes: $\bar{P}_k^G = (0, 2kW)$, Substation node: \bar{P}_1^G scaled with n . At all nodes, $\underline{P}_k^G = 0$.
Reactive power gen. limits	$\bar{Q}_k^G = 0.3\bar{P}_k^G, \underline{Q}_k^G = -0.3\bar{P}_k^G$ at all nodes.	$\bar{Q}_k^G = 0.3\bar{P}_k^G, \underline{Q}_k^G = -0.3\bar{P}_k^G$ at all nodes.
Base quantities	$P_{base} = 1MW, V_{base} = 12.35kV(L - L)$.	$P_{base} = 1MW, V_{base} = 12.47kV(L - L)$.

TABLE I: Circuit Parameters for SDR of OPF

Test system	SoCal distribution circuit		Random radial networks	
Minimize	Power-loss	Voltage	Power-loss	Voltage
rank W_*	1	≥ 1	1	≥ 1
Mean η	N/A	1.8%	N/A	0.5%
Maximum η	N/A	4.1%	N/A	1.5%

TABLE II: Summary of simulation results

finds a feasible point of OPF with an objective value close to the theoretical optimum and hence performs well. A general bound on the performance of this heuristic technique, however, remains an open question.

V. CONCLUSION

QCQP problems are generally non-convex and NP-hard. This paper proves that a certain class of QCQP problems are solvable in polynomial-time. We have applied this result to the optimal power flow problem and derived a set of conditions under which this nonconvex problem admits an efficient solution. For problems that do not satisfy our sufficient conditions, we provide a heuristic technique to find a feasible solution. Simulations suggest that this method often finds a near-optimal solution for the OPF problem.

VI. APPENDIX

A. Proof of lemma 2:

The result is restated here for convenience: If $H_1 \succeq 0$ and $H_2 \succeq 0$ are two $n \times n$ matrices,

$$\text{tr}(H_1 H_2) \geq \rho_{\min}[H_1] \rho_{\max}[H_2].$$

Proof: Let the spectral decomposition of H_2 be

$$H_2 = \sum_{i=1}^n \rho_i u_i u_i^{\mathcal{H}},$$

where $\|u_i\| = 1$ for all $1 \leq i \leq n$. Then:

$$\begin{aligned} \text{tr}(H_1 H_2) &= \sum_{i=1}^n \rho_i (u_i^{\mathcal{H}} H_1 u_i) \\ &\geq \sum_{i=1}^n \rho_i \rho_{\min}[H_1] \\ &\geq \rho_{\min}[H_1] \rho_{\max}[H_2]. \end{aligned}$$

■

B. Proof of Theorem 1 for $C \succeq 0$:

The sketch of the proof is as follows. Perturb RP so that the matrix in the objective function is positive definite. From our previous analysis this perturbed problem has a finite optimizer that has rank at most 1. Also, this perturbed problem can be solved in polynomial time. Using the solutions from the perturbed problems, we construct an optimum solution of P in polynomial time.

In particular, consider the perturbed problems for $\delta > 0$:

Perturbed primal problem $P^{(\delta)}$:

$$\begin{aligned} &\underset{x \in \mathbb{C}^n}{\text{minimize}} && x^{\mathcal{H}}(C + \delta \mathcal{I})x \\ &\text{subject to:} && x^{\mathcal{H}} C_k x \leq b_k, \quad k \in \mathcal{K}. \end{aligned}$$

Perturbed relaxed problem $RP^{(\delta)}$:

$$\begin{aligned} &\underset{W \succeq 0}{\text{minimize}} && \text{tr}[(C + \delta \mathcal{I})W] \\ &\text{subject to:} && \text{tr}(C_k W) \leq b_k, \quad k \in \mathcal{K}, \end{aligned}$$

where \mathcal{I} is the $n \times n$ identity matrix. For any variable z in P/RP , let $z^{(\delta)}$ denote the corresponding variable in $P^{(\delta)}/RP^{(\delta)}$.

The matrix $C + \delta\mathcal{I}$ is positive definite for all $\delta > 0$. There exists $W_*^{(\delta)} \succeq 0$ that solves $RP^{(\delta)}$ and $\text{rank } W_*^{(\delta)} \leq 1$. Let the spectral decomposition of $W_*^{(\delta)}$ be $W_*^{(\delta)} = (x_*^{(\delta)})(x_*^{(\delta)})^H$. From Lemma 3, we have

$$p_*^{(\delta)} = (x_*^{(\delta)})^H (C + \delta\mathcal{I}) (x_*^{(\delta)}) = r_*^{(\delta)} = \text{tr}[(C + \delta\mathcal{I})W_*^{(\delta)}].$$

The feasible regions of P and $P^{(\delta)}$ are the same and are bounded. Then $x_*^{(\delta)}$ lies in a compact space, independent of δ . Taking a sequence $\delta \rightarrow 0$, the corresponding sequence of $x_*^{(\delta)}$ has a convergent subsequence. Let the limit point of this subsequence be \hat{x} . Then \hat{x} is feasible for P . We now show that \hat{x} solves P optimally.

The objective function of $P^{(\delta)}$ increases with δ . Therefore,

$$p_* \leq \hat{x}^H C \hat{x} \leq p_*^{(\delta)} = r_*^{(\delta)}. \quad (23)$$

We wish to show that the first inequality is an equality. Suppose on the contrary $p_* < \hat{x}^H C \hat{x}$. Let x'_* be any finite optimizer of P . Then,

$$p_* = (x'_*)^H C (x'_*) < \hat{x}^H C \hat{x}, \quad (24)$$

and we can choose a sufficiently small $\delta > 0$, such that

$$p_* + \delta(x'_*)^H C (x'_*) < \hat{x}^H C \hat{x} \leq r_*^{(\delta)}.$$

This follows from equations (23) and (24). For this δ , $[(x'_*)(x'_*)^H + (x_*^{(\delta)})(x_*^{(\delta)})^H]/2$ is a feasible point of $RP^{(\delta)}$ and satisfies

$$\begin{aligned} r_*^{(\delta)} &\leq \text{tr} \left[(C + \delta\mathcal{I}) \left(\frac{(x'_*)(x'_*)^H + (x_*^{(\delta)})(x_*^{(\delta)})^H}{2} \right) \right] \\ &= \frac{1}{2} r_*^{(\delta)} + \frac{1}{2} [p_* + \delta(x'_*)^H C (x'_*)] \\ &< r_*^{(\delta)}. \end{aligned}$$

This is a contradiction and hence $p_* = \hat{x}^H C \hat{x}$.

Now, we show how to use this perturbation technique to solve P in polynomial time. Solve RP to get $p_* = r_*$. If the optimizer W_* of RP has rank at most 1, compute x_* from W_* as in lemma 3 then we have solved P in polynomial time. If it does not satisfy the rank condition,

then choose an arbitrary $\delta_0 > 0$ and solve $RP^{(\delta_0)}$ in polynomial time to get the minimizer $W_*^{(\delta_0)}$ and the minimum $r_*^{(\delta_0)} = p_*^{(\delta_0)}$. For any δ in $(0, \delta_0)$,

$$p_* = r_* \leq \text{tr}(CW_*^{(\delta)}) \leq p_*^{(\delta)}. \quad (25)$$

Also, $p_*^{(\delta)}$ is convex in δ [41] and hence

$$p_*^{(\delta)} \leq p_* + \frac{\delta}{\delta_0} (p_*^{(\delta_0)} - p_*). \quad (26)$$

Given $\zeta > 0$, choose δ sufficiently small so that $\frac{\delta}{\delta_0} (p_*^{(\delta_0)} - p_*) \leq \zeta$. For this δ , solve $RP^{(\delta)}$ arbitrarily closely in polynomial time to get $W_*^{(\delta)}$ that has rank at most 1 and compute $x_*^{(\delta)}$. From equations (25) and (26), $x_*^{(\delta)}$ satisfies

$$|(x_*^{(\delta)})^H C(x_*^{(\delta)}) - p_*| \leq \zeta.$$

Thus $x_*^{(\delta)}$ is a feasible point of P that has a value of the objective function within ζ of the theoretical optimum and we have shown a polynomial time algorithm to compute it.

This completes the proof of Theorem 1.

C. Matrices involved in OPF:

Here we compute the (i, j) -th entries of $\{C_k, k \in \mathcal{K}\}$ for OPF . From (14), (15), (16), we have the following relations for $k \in [n]$, and (p, q) and (i, j) in \mathcal{T} :

$$[\Phi_k]_{ij} = \begin{cases} \frac{1}{2}Y_{ij} = \frac{1}{2}(-g_{ij} + \mathbf{i}b_{ij}) & \text{if } k = i \\ \frac{1}{2}Y_{ij}^H = \frac{1}{2}(-g_{ij} - \mathbf{i}b_{ij}) & \text{if } k = j \\ 0 & \text{otherwise,} \end{cases} \quad (27)$$

$$[\Psi_k]_{ij} = \begin{cases} \frac{-1}{2\mathbf{i}}Y_{ij} = \frac{1}{2}(-b_{ij} - \mathbf{i}g_{ij}) & \text{if } k = i \\ \frac{1}{2\mathbf{i}}Y_{ij}^H = \frac{1}{2}(-b_{ij} + \mathbf{i}g_{ij}) & \text{if } k = j \\ 0 & \text{otherwise,} \end{cases} \quad (28)$$

$$[M^{pq}]_{ij} = \begin{cases} g_{pq} & \text{if } i = j = p \\ \frac{1}{2}(-g_{pq} + \mathbf{i}b_{pq}) & \text{if } (i, j) = (p, q) \\ \frac{1}{2}(-g_{pq} - \mathbf{i}b_{pq}) & \text{if } (i, j) = (q, p) \\ 0 & \text{otherwise,} \end{cases} \quad (29)$$

$$[T^{pq}]_{ij} = \begin{cases} g_{pq} & \text{if } i = j = p \text{ or } i = j = q \\ -g_{pq} & \text{if } (i, j) = (p, q) \text{ or } (i, j) = (q, p) \\ 0 & \text{otherwise.} \end{cases} \quad (30)$$

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